ON THE STRUCTURE OF COMPLEX HOMOGENEOUS SUPERMANIFOLDS

E.G. Vishnyakova [1]

Tver State University

Abstract. For a Lie group G and a closed Lie subgroup $H \subset G$, it is well known that the coset space G/H can be equipped with the structure of a manifold homogeneous under G and that any G-homogeneous manifold is isomorphic to one of this kind. An interesting problem is to find an analogue of this result in the case of supermanifolds.

In the classical setting, G is a real or a complex Lie group and G/H is a real and, respectively, a complex manifold. Now, if G is a real Lie supergroup and $H \subset G$ is a closed Lie subsupergroup, there is a natural way to consider G/H as a supermanfold. Furthermore, any G-homogeneous real supermanifold can be obtained in this way, see [1]. The goal of this paper is to give a proof of this result in the complex case.

1. Preliminaries

We will use the word "supermanifold" in the sense of Beresin-Leĭtes-Kostant (see [1, 2]). All the time, we will be interested in the complex-analytic version of the theory. A morphism $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between two supermanifolds is denoted by $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1 : M \to N$ is a continuous mapping and $\varphi_2 : \mathcal{O}_N \to (\varphi_1)_*(\mathcal{O}_M)$ is a homomorphism of sheaves of superalgebras. We begin with the more general notion of a Lie supergroup.

A *Lie supergroup* is a supermanifold $(G, \mathcal{O}_G)$, for which the following three morphisms are defined: $\nu : (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ (multiplication morphism), $\iota : (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ (passing to the inverse), $\varepsilon : (\mathrm{pt}, \mathbb{C}) \to (G, \mathcal{O}_G)$ (identity morphism). Moreover, these morphisms should satisfy the usual conditions, modeling the group axioms. The underlying manifold G is a Lie group. We denote by $\mathfrak{g}$ the Lie superalgebra of $(G, \mathcal{O}_G)$ (see [2] for the corresponding definition).

An *action of a Lie supergroup* $(G, \mathcal{O}_G)$ *on a supermanifold* $(M, \mathcal{O}_M)$ is a morphism $\mu : (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$, such that the following conditions hold:

- $\mu \circ (\nu \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \mu)$;
- $\mu \circ (\varepsilon \times \mathrm{id}) = \mathrm{id}$.

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We denote by $\mathfrak{v}(M,\mathcal{O}_M)$ the Lie superalgebra of holomorphic vector fields on $(M,\mathcal{O}_M)$. Let $\mathfrak{m}_x$ be the maximal ideal of the local superalgebra $(\mathcal{O}_M)_x$. The vector superspace $T_x(M,\mathcal{O}_M) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the *tangent space* to $(M,\mathcal{O}_M)$ at $x \in M$. From the inclusions $v(\mathfrak{m}_x) \subset (\mathcal{O}_M)_x$ and $v(\mathfrak{m}_x^2) \subset \mathfrak{m}_x$, where $v \in \mathfrak{v}(M,\mathcal{O}_M)$, it follows that there exists an even linear mapping $\mathrm{ev}_x(v) : \mathfrak{m}_x/\mathfrak{m}_x^2 \to (\mathcal{O}_M)_x/\mathfrak{m}_x \simeq \mathbb{C}$. In other words, $\mathrm{ev}_x(v) \in T_x(M,\mathcal{O}_M)$, and so we obtain a map $\mathrm{ev}_x : \mathfrak{v}(M,\mathcal{O}_M) \to T_x(M,\mathcal{O}_M)$.

Let $(U,\mathcal{O}_M) \subset (M,\mathcal{O}_M)$ be a superdomain with even and, respectively, odd coordinates (x_i) and (ξ_j) and let $f \in \mathcal{O}_M(U)$. We can write f in the form

$$f = f_0 + \sum_i f_i\xi_i + \sum_{ij} f_{ij}\xi_i\xi_j + \dots,$$

where $f_{ij\dots}$ are some holomorphic functions on U. For $p \in U$ we will denote by $f(p)$ the value of f_0 at p.

Let $X \in T_x(M,\mathcal{O}_M)$. There is a neighborhood $(U,\mathcal{O}_M)$ of the point x and a vector field $v_X \in \mathfrak{v}(U,\mathcal{O}_M)$ such that $\mathrm{ev}_x(v_X) = X$. We can consider X as a linear function on $(\mathcal{O}_M)_x$. Namely, $X(f_x) := (v_X(f_x))(x)$, where $f_x \in (\mathcal{O}_M)_x$.

Let $\mu = (\mu_1,\mu_2) : (G,\mathcal{O}_G) \times (M,\mathcal{O}_M) \to (M,\mathcal{O}_M)$ be an action. Then there is a homomorphism of the Lie superalgebras $\widetilde{\mu} : \mathfrak{g} \to \mathfrak{v}(M,\mathcal{O}_M)$, given by the formula $X \mapsto (\varepsilon \times \mathrm{id})_2 \circ (X \oplus 0) \circ \mu_2$, where $X \in \mathfrak{g}$. An action μ is called *transitive* if the mapping $\mathrm{ev}_x \circ \widetilde{\mu}$ is surjective for all $x \in M$, see [4]. In this case the supermanifold $(M,\mathcal{O}_M)$ is called $(G,\mathcal{O}_G)$-*homogeneous*. A supermanifold $(M,\mathcal{O}_M)$ is called *homogeneous*, if it possesses a transitive action of some Lie supergroup.

Let $\phi = (\phi_1,\phi_2) : (M,\mathcal{O}_M) \to (M_1,\mathcal{O}_{M_1})$ be a morphism of supermanifolds. Denote by $(\mathrm{d}\phi)_x$ the differential of ϕ at $x \in M$ (see [2, 3]). Let $\dim(M,\mathcal{O}_M) = n|m$, $\dim(M_1,\mathcal{O}_{M_1}) = k|l$. The morphism $\phi : (M,\mathcal{O}_M) \to (M_1,\mathcal{O}_{M_1})$ is called a *submersion at* $p \in M$ if $n \geq k$, $m \geq l$ and there exist two neighborhoods $(U,\mathcal{O}_M)$ of p and $(V,\mathcal{O}_{M_1})$ of $q = \phi_1(p)$ with the coordinates $(x_i,\ \xi_j)$ and, respectively, $(y_s,\ \eta_t)$ such that $\phi_2|(U,\mathcal{O}_M)$ is given by the formulas:

$$\phi_2(y_i) = x_i,\ i = 1,\dots,k, \quad \phi_2(\eta_j) = \xi_j,\ j = 1,\dots,l. \tag{1}$$

This definition is equivalent to the requirement that the mapping $(\mathrm{d}\phi)_p$ is surjective (see [2, 3]). A morphism $\phi = (\phi_1,\phi_2) : (M,\mathcal{O}_M) \to (M_1,\mathcal{O}_{M_1})$ is called an *immersion at* $p \in M$, if $n < k$, $m < l$, and there are neighborhoods $(U,\mathcal{O}_M)$ and $(V,\mathcal{O}_{M_1})$ of p and $q = \phi_1(p)$ with coordinates $(x_i,\ \xi_j)$ and, respectively, $(y_s,\ \eta_t)$, such that the morphism $\phi_2|(U,\mathcal{O}_M)$ is given by the formulas:

$$\begin{aligned} &\phi_2(y_i) = x_i,\ i = 1,\dots,m, \quad \phi_2(y_i) = 0,\ i > m,\\ &\phi_2(\eta_j) = \xi_j,\ j = 1,\dots,n, \quad \phi_2(\eta_j) = 0,\ j > n. \end{aligned} \tag{2}$$

This definition is equivalent to the requirement that the mapping $(\mathrm{d}\phi)_p$ is injective (see [2, 3]).

Let $(M,\mathcal{O}_M)$ be a supermanifold. A supermanifold $(U,\mathcal{O}_M)$, where U is an open subset in M, is called an *open subsupermanifold* of $(M,\mathcal{O}_M)$. Suppose that $M \subset M_1$ is a topological subspace and denote by $\phi_1 : M \to M_1$ the embedding. A supermanifold $(M,\mathcal{O}_M)$ is called a *subsupermanifold* of a supermanifold $(M_1,\mathcal{O}_{M_1})$ if there is a morphism $\phi : (M,\mathcal{O}_M) \to (M_1,\mathcal{O}_{M_1})$, such that the differential $(\mathrm{d}\phi)_p$ is injective at every point $p \in M$ and the first component of ϕ coincides with ϕ_1. In this case we will use the notation $(M,\mathcal{O}_M) \subset (M_1,\mathcal{O}_{M_1})$.

Let $(K,\mathcal{O}_K) \subset (M,\mathcal{O}_M)$ be a subsupermanifold, $\mathcal{I} \subset \mathcal{O}_M$ the sheaf of ideals corresponding to $(K,\mathcal{O}_K)$, and $\mathcal{J} \subset \mathcal{O}_{G\times M}$ the sheaf of ideals corresponding to $(G \times K,\mathcal{O}_{G\times K}) \subset (G \times M,\mathcal{O}_{G\times M})$ (see [2]). We will say that $(K,\mathcal{O}_K)$ is μ-*invariant* if the following conditions hold:

1. $\mu_1(G,K) \subset K$,

2. $\mu_2(\mathcal{I}) \subset (\mu_1)_*\mathcal{J}$.

A *Lie subsupergroup* of a Lie supergroup $(G,\mathcal{O}_G)$ is a subsupermanifold $(H,\mathcal{O}_H) \subset (G,\mathcal{O}_G)$, such that $e \in H$ and $(H,\mathcal{O}_H)$ is ν- and ι-invariant.

If $\phi : (M,\mathcal{O}_M) \to (M_1,\mathcal{O}_{M_1})$ is a morphism of supermanifolds and $(N,\mathcal{O}_N) \subset (M,\mathcal{O}_M)$ is a subsupermanifold, we denote by $\phi|_N$ the composition:

$$(N,\mathcal{O}_N) \hookrightarrow (M,\mathcal{O}_M) \xrightarrow{\Phi} (M_1,\mathcal{O}_{M_1}).$$

Let $(G,\mathcal{O}_G)$ be a Lie supergroup. To each point $g \in G$ assign a morphism $\widehat{g} = (\widehat{g}_1,\widehat{g}_2) : (\mathrm{pt},\mathbb{C}) \to (G,\mathcal{O}_G)$. Namely, let $\widehat{g}_1(\mathrm{pt}) = g$ and define $\widehat{g}_2 : \mathcal{O}_G \to (\widehat{g}_1)_*(\mathbb{C})$ by $\widehat{g}_2(f_x) = 0$ for $f_x \in (\mathcal{O}_G)_x$ if $x \neq g$ and $\widehat{g}_2(f_g) = f_g(g)$ for $f_g \in (\mathcal{O}_G)_g$. Denote by l_g, $g \in G$, the composition of the morphisms

$$(G,\mathcal{O}_G) \xrightarrow{\sim} (\mathrm{pt},\mathbb{C}) \times (G,\mathcal{O}_G) \xrightarrow{\widehat{g}\times id} (G,\mathcal{O}_G) \times (G,\mathcal{O}_G) \xrightarrow{\nu} (G,\mathcal{O}_G).$$

and define r_g in a similar way. Since there exists the inverse morphism $l_{g^{-1}}$, the morphism l_g is an automorphism of the supermanifold $(G,\mathcal{O}_G)$, and the same is true for r_g. We refer the reader to [2, 3] for the definition of a superdomain and the proof of the following inverse function theorem.

Theorem 1. *Let $(U,\mathcal{O}_U)$ and $(V,\mathcal{O}_V)$ be two superdomains with coordinate systems (x_i,ξ_j) and (y_s,η_t). Let $\phi : (U,\mathcal{O}_U) \to (V,\mathcal{O}_V)$ be a morphism and let $u \in U$. Then the following conditions are equivalent:*

- *ϕ is an isomorphism in some neighborhood of u;*
- *$(\mathrm{d}\phi)_u$ is an isomorphism.*

Given supermanifolds $(M_i,\mathcal{O}_{M_i})$, $i = 1,\dots,n$, we will denote by $\mathrm{pr}^{M_1\times\cdots\times M_n}_{M_i}$ the natural projection $(M_1,\mathcal{O}_{M_1}) \times \cdots \times (M_n,\mathcal{O}_{M_n}) \to (M_i,\mathcal{O}_{M_i})$.

2. The structure of a supermanifold on G/H

Let $(K,\mathcal{O}_K) \subset (U,\mathcal{O}_G|_U)$ be a subsupermanifold of an open subsupermanifold $(U,\mathcal{O}_G|_U)$ in a Lie supergroup $(G,\mathcal{O}_G)$, $\mathcal{I}_K$ the corresponding sheaf of ideals and φ is an isomorphism of $(G,\mathcal{O}_G)$. We will denote by $(\varphi(K),\mathcal{O}_{\varphi(K)})$ the subsupermanifold of $(\varphi(U),\mathcal{O}_G|_{gU})$, where $\varphi(K) := \varphi_1(K)$, $\mathcal{O}_{\varphi(K)} := (\mathcal{O}_G|_{\varphi(U)})/(\varphi^{-1})_2(\mathcal{I}_K)$. Sometimes we will use the notation $(gK,\mathcal{O}_{gK})$ for $(l_g(K),\mathcal{O}_{l_g(K)})$. The subsupermanifold $(Kg,\mathcal{O}_{Kg})$ is defined analogously. The following proposition is well known.

ł *Let $\varphi = (\varphi_1,\varphi_2) : (M,\mathcal{O}_M) \to (N,\mathcal{O}_N)$ be a morphism of supermanifolds. Assume that $\varphi_1 : M \to N$ is a homeomorphism and φ is a local isomorphism. Then φ is an isomorphism.*
$\square$

Let $(N, \mathcal{O}_N)$ and $(S, \mathcal{O}_S)$ be two subsupermanifolds of $(M, \mathcal{O}_M)$. The subsupermanifold $(S, \mathcal{O}_S)$ is called *transversal to the subsupermanifold* $(N, \mathcal{O}_N)$ *at a point* $x \in S \cap N$, if $T_x(M, \mathcal{O}_M) = T_x(N, \mathcal{O}_N) \oplus T_x(S, \mathcal{O}_S)$.

Theorem 2. *Let $(G, \mathcal{O}_G)$ be a Lie supergroup and $(H, \mathcal{O}_H)$ be a Lie subsupergroup of $(G, \mathcal{O}_G)$. Suppose that $(S', \mathcal{O}_{S'})$ is a transversal subsupermanifold to $(H, \mathcal{O}_H)$ at the point e (the identity element of G). Then there is a subsupermanifold $(S, \mathcal{O}_S)$, such that $\nu|_{S\times H} : (S, \mathcal{O}_S) \times (H, \mathcal{O}_H) \to (G, \mathcal{O}_G)$ is an isomorphism of $(S, \mathcal{O}_S) \times (H, \mathcal{O}_H)$ onto an open subsupermanifold $(U, \mathcal{O}_G) \subset (G, \mathcal{O}_G)$.*

Proof. Let $\dim(G, \mathcal{O}_G) = n|m$. First we will show that there exists a transversal subsupermanifold to $(H, \mathcal{O}_H)$ at e. From the definition of a subsupermanifold it follows that there is a superdomain $(V, \mathcal{O}_V) \subset (G, \mathcal{O}_G)$ containing e, such that the subsupermanifold $(H, \mathcal{O}_H)$ is given by the equations $x_i = 0$, $i \in \Gamma_1$, and $\xi_j = 0$, $j \in \Gamma_2$, where $\Gamma_1 \subset \{1, \dots, n\}$, $\Gamma_2 \subset \{1, \dots, m\}$. Denote by $(S', \mathcal{O}_{S'})$ the subsupermanifold in $(V, \mathcal{O}_V)$ given by the equations $x_i = 0$, $i \notin \Gamma_1$, and $\xi_j = 0$, $j \notin \Gamma_2$. Obviously, $(S', \mathcal{O}_{S'})$ is transversal to $(H, \mathcal{O}_H)$.

Since $\nu|_{S\times H} \circ (\varepsilon \times \mathrm{id}) = \mathrm{id}$ and $\nu|_{S\times H} \circ (\mathrm{id} \times \varepsilon) = \mathrm{id}$, it follows that the differential $(\mathrm{d}\,\nu|_{S\times H})_{(e,e)} : T_e(S_1, \mathcal{O}_{S_1}) \oplus T_e(H, \mathcal{O}_H) = T_e(G, \mathcal{O}_G) \to T_e(G, \mathcal{O}_G)$ is precisely the mapping

$$(v_s, v_h) \mapsto v_s + v_h. \tag{3}$$

Now it is easy to see that $(\mathrm{d}\,\nu|_{S\times H})_{(e,e)}$ is an isomorphism. From Theorem 1, it follows that there are superdomains $(S, \mathcal{O}_S) \subset (S', \mathcal{O}_{S'})$, $(H', \mathcal{O}_H) \subset (H, \mathcal{O}_H)$ and $(W, \mathcal{O}_G) \subset (G, \mathcal{O}_G)$, such that $\nu|_{S\times H'} : (S, \mathcal{O}_S) \times (H', \mathcal{O}_H) \to (W, \mathcal{O}_G)$ is an isomorphism.

Fix a point $h \in H$. By the associativity axiom for Lie supergroups we have the following commutative diagram:

$$\begin{array}{ccc} (S, \mathcal{O}_S) \times (H', \mathcal{O}_H) & \xrightarrow{id\times r_h|_H} & (S, \mathcal{O}_S) \times (H'h, \mathcal{O}_H) \\ {\scriptstyle \nu|_{S\times H'}}\big\downarrow & & \big\downarrow{\scriptstyle \nu|_{S\times H'h}} \\ (G, \mathcal{O}_G) & \xrightarrow{r_h} & (G, \mathcal{O}_G) \end{array}.$$

In other words, the morphism ν, restricted to $(S, \mathcal{O}_S) \times (H'h, \mathcal{O}_H)$, is equal to $r_h \circ \nu|_{S\times H'} \circ (id \times r_h|_H)^{-1}$. We have

$$(S, \mathcal{O}_S) \times (H, \mathcal{O}_H) = \bigcup_{h\in H} \big\{(S, \mathcal{O}_S) \times (H'h, \mathcal{O}_H)\big\}.$$

Therefore $\nu|_{S\times H}$ is a local isomorphism. By a well known argument from the geometric theory of homogeneous spaces we may assume that $(\nu|_{S\times H})_1 : S\times H \to G$ is a homeomorphism. Using Lemma 1, we get that $\nu|_{S\times H} : (S, \mathcal{O}_S) \times (H, \mathcal{O}_H) \to (U, \mathcal{O}_U)$ is an isomorphism. $\square$

Now we will give the definition of a supermanifold with the underlying manifold G/H, corresponding to a Lie supergroup $(G, \mathcal{O}_G)$ and a subsupergroup $(H, \mathcal{O}_H) \subset (G, \mathcal{O}_G)$. Let $p_1 : G \to G/H$, $g \mapsto gH$ be the natural mapping. Fix a transversal subsupermanifold $(S, \mathcal{O}_S) \subset (G, \mathcal{O}_G)$ to $(H, \mathcal{O}_H)$ at the point e so that $\nu|_{S\times H}$ is an isomorphism (see Theorem 2). The mapping p_1 maps S homeomorphically onto a domain $V \subset G/H$. Denote by $\mathcal{O}_V$ the sheaf $(p_1)_*(\mathcal{O}_S)$ and identify $(S, \mathcal{O}_S)$ with $(V, \mathcal{O}_V)$. Recall that the open subsupermanifold

$(U,\mathcal{O}_G)\subset(G,\mathcal{O}_G)$ is defined in Theorem 2. Let $p_U=(p_1,(p_U)_2)$ be such a morphism that the following diagram is commutative:

$$\begin{array}{ccc} (S,\mathcal{O}_S)\times(H,\mathcal{O}_H) & \xrightarrow{\nu|_{S\times H}} & (U,\mathcal{O}_G) \\ {\scriptstyle \mathrm{pr}_V^{S\times H}}\downarrow & & \downarrow{\scriptstyle p_U} \\ (V,\mathcal{O}_V) & = & (V,\mathcal{O}_V) \end{array}.$$

We can define $(gV,\mathcal{O}_{gV})$ analogously. Namely, it is clear that $\nu|_{gS\times H}:(gS,\mathcal{O}_{gS})\times(H,\mathcal{O}_H)\to(gU,\mathcal{O}_{gU})$ is an isomorphism and $p_1:gS\to gV$ is a homeomorphism. Denote by $\mathcal{O}_{gV}$ the sheaf $(p_1)_*(\mathcal{O}_{gS})$ on gV and identify the subsupermanifold $(gS,\mathcal{O}_{gS})$ with $(gV,\mathcal{O}_{gV})$. Define a morphism $p_{gU}:(gU,\mathcal{O}_G)\to(gV,\mathcal{O}_{gV})$ by

$$p_{gU}=\mathrm{pr}_{gV}^{gS\times H}\circ(\nu|_{gS\times H})^{-1}.$$

Now we need the following lemma.

ł *Let $(W,\mathcal{O}_W)$ be a superdomain, $(F,\mathcal{O}_F)$ a supermanifold, and $(M,\mathcal{O}_M)=(W,\mathcal{O}_W)\times(F,\mathcal{O}_F)$. Denote by (x_i) all, i.e. even and odd, coordinates on $(W,\mathcal{O}_W)$. Then any function $f\in\mathcal{O}_M(M)$ can be written in the form $f=\sum_j h_jg_j$, where (h_j) is a maximal independent system of polynomials in x_i and $g_j\in\mathcal{O}_F(F)$.*

Proof. Fix a coordinate neighborhood $(N,\mathcal{O}_F)$ of the supermanifold $(F,\mathcal{O}_F)$. The function $f|_{W\times N}$ has the form $\sum_j h_jt_j^N$, where $t_j^N\in\mathcal{O}_F(N)$. Let $(\tilde{N},\mathcal{O}_F)$ be another coordinate neighborhood of $(F,\mathcal{O}_F)$, such that $N\cap\tilde{N}\neq\varnothing$. Then the function $f|_{W\times\tilde{N}}$ has the form

$$f=\sum_j h_jt_j^{\tilde{N}},$$

where $t_j^{\tilde{N}}\in\mathcal{O}_F(\tilde{N})$. It is obvious that $t_j^N=\tilde{t}_j^{\tilde{N}}$ in $N\cap\tilde{N}$. Now we choose an atlas $\{(N,\mathcal{O}_F)\}$ on the supermanifold $(F,\mathcal{O}_F)$. The functions t_j^N are holomorphic in all coordinate neighborhoods of the chosen atlas and coincide on their intersections. It follows that there are $g_j\in H^0(F,\mathcal{O}_F)$ such that $g_j|_N=t_j^N$, and so we can write $f=\sum_j h_jg_j$.□

Let $W\subset G/H$ be an open set. A function $f\in\mathcal{O}_G(p_1^{-1}(W))$ is called $(H,\mathcal{O}_H)$*-right invariant* if $(\nu|_{G\times H})_2(f)=(\mathrm{pr}_{p_1^{-1}(W)}^{p_1^{-1}(W)\times H})_2(f)$ (see [1]).

ł *Let W be an open set in gV, $f\in\mathcal{O}_G(p_{gU}^{-1}(W))$. Then $f\in\mathrm{Im}(p_{gU})_2$ if and only if f is a $(H,\mathcal{O}_H)$-right invariant function.*

Proof. By construction, we have $\mathrm{Im}(p_{gU})_2=((\nu|_{gS\times H})^{-1})_2((\mathrm{pr}_{gS}^{gS\times H})_2(\mathcal{O}_{gS}))$. By associativity of the multiplication in $(G,\mathcal{O}_G)$, the following diagram is commutative:

$$\begin{array}{ccc} (gU,\mathcal{O}_G)\times(H,\mathcal{O}_H) & \xrightarrow{\nu|_{G\times H}} & (gU,\mathcal{O}_G) \\ {\scriptstyle \nu|_{gS\times H}\times\mathrm{id}}\uparrow & & \uparrow{\scriptstyle \nu|_{gS\times H}} \\ (gS,\mathcal{O}_{gS})\times(H,\mathcal{O}_H)\times(H,\mathcal{O}_H) & \xrightarrow{\mathrm{id}\times\nu|_{H\times H}} & (gS,\mathcal{O}_{gS})\times(H,\mathcal{O}_H) \end{array}. \qquad (4)$$

Now we can see that if $f \in \operatorname{Im}(p_{gU})_2$ then f is a $(H, \mathcal{O}_H)$-right invariant function. Indeed, in this case $(\nu|_{gS\times H})_2(f) \in (\mathrm{pr}_{gS}^{gS\times H})_2(\mathcal{O}_{gS})$ and

$$(\mathrm{id}_2 \times (\nu|_{H\times H})_2)((\nu|_{gS\times H})_2(f)) = (\mathrm{pr}_{gS}^{gS\times H\times H})_2((\nu|_{gS\times H})_2(f)).$$

Applying the isomorphism $(\nu|_{gS\times H} \times \mathrm{id})^{-1}$ to the right hand side, we get:

$$(\nu|_{gS\times H} \times \mathrm{id})_2^{-1} \circ (\mathrm{pr}_{gS}^{gS\times H\times H})_2((\nu|_{gS\times H})_2(f)) = (\mathrm{pr}_{gU}^{gU\times H})_2(f).$$

By the commutativity of (4), we obtain:

$$(\nu|_{G\times H})_2(f) = (\mathrm{pr}_{p_1^{-1}(W)}^{p_1^{-1}(W)\times H})_2(f).$$

Conversely, let $f \in \mathcal{O}_G(p_{gU}^{-1}(W))$ be a $(H, \mathcal{O}_H)$-right invariant function. Without loss of generality assume that $(gS, \mathcal{O}_{gS})$ is a coordinate neighborhood. In the present setting, there will be no confusion to denote the even and odd coordinates by the same letters (x_i). By Lemma 2, we have

$$(\nu|_{gS\times H})_2(f) = \sum h_i g_i,$$

where (h_i) is a maximal independent system of monomials in x_i, $g_i \in \mathcal{O}_H(H)$. By the definition of a $(H, \mathcal{O}_H)$-right invariant function and the commutativity of (4), we get

$$(\mathrm{id} \times \nu|_{H\times H})_2(\sum h_i g_i) = (\mathrm{pr}_{gS\times H}^{gS\times H\times H})_2(\sum h_i g_i) = \sum (\mathrm{pr}_{gS}^{gS\times H\times H})_2(h_i)(\mathrm{pr}_{H_1}^{gS\times H\times H})_2(g_i),$$

where $\mathrm{pr}_{H_1}^{gS\times H\times H}$ is the projection onto the second factor. On the other hand,

$$(\mathrm{id} \times \nu|_{H\times H})_2(\sum h_i g_i) = \sum (\mathrm{pr}_{gS}^{gS\times H\times H})_2(h_i)(\mathrm{pr}_{H\times H}^{gS\times H\times H})_2((\nu|_{H\times H})_2(g_i)).$$

The independence of (h_i) implies that $(\mathrm{pr}_{H_1}^{H\times H})_2(g_i) = (\nu|_{H\times H})_2(g_i)$, where $\mathrm{pr}_{H_1}^{H\times H}$ is the projection onto the first factor. Equivalently, the functions g_i are $(H, \mathcal{O}_H)$-right invariant. We have reduced our assertion to the following one.

$(*)$ *If a function $g \in \mathcal{O}_H(H)$ is $(H, \mathcal{O}_H)$-right invariant, then $g = const$.*

Proof of $(*)$. As above, let $\mathrm{pr}_{H_1}^{H\times H} : (H, \mathcal{O}_H) \times (H, \mathcal{O}_H) \to (H, \mathcal{O}_H)$ denote the projection onto the first factor. By the definition of a $(H, \mathcal{O}_H)$-right invariant function, we have $(\mathrm{pr}_{H_1}^{H\times H})_2(g) = (\nu|_{H\times H})_2(g)$. Now we get

$$g(e) = (\varepsilon \times \mathrm{id})_2((\mathrm{pr}_{H_1}^{H\times H})_2(g)) = (\varepsilon \times \mathrm{id})_2((\nu|_{H\times H})_2(g)) = g,$$

where the last equality follows from the identity axiom $\nu \circ (\varepsilon \times \mathrm{id}) = \mathrm{id}$. Therefore $g = g(e)$ showing $(*)$. This completes the proof of Lemma 2. □

Theorem 3. *The charts $(gV, \mathcal{O}_{gV})$ constitute a holomorphic atlas on G/H.*

Proof. Suppose that $g_1V \cap g_2V \neq \emptyset$. Let us prove that there is a morphism $\Psi_{g_1V,g_2V} : (g_1V \cap g_2V, \mathcal{O}_{g_1V}) \to (g_1V \cap g_2V, \mathcal{O}_{g_2V})$ such that $p_{g_2U} = \Psi_{g_1V,g_2V} \circ p_{g_1U}$. Obviously, $(\Psi_{g_1V,g_2V})_1 = \mathrm{id}$. Let us define the second component $(\Psi_{g_1V,g_2V})_2$.

If $f \in \mathcal{O}_{g_2V}|_{g_1V\cap g_2V}$ then $(p_{g_2U})_2(f) \in \mathcal{O}_G|_{g_1U\cap g_2U}$. By Lemma 2, the function $(p_{g_2U})_2(f)$ is $(H, \mathcal{O}_H)$-right invariant. Therefore $(p_{g_2U})_2(f) \in \operatorname{Im}((p_{g_1U})_2)$. By construction, the map

$(p_{g_1U})_2$ is injective, and so there is a unique function $g \in \mathcal{O}_{g_1V}|_{g_1V\cap g_2V}$, such that $(p_{g_1U})_2(g) = (p_{g_2U})_2(f)$. We put $(\Psi_{g_1V,g_2V})_2(f) := g$. The cocycle condition is obviously fulfilled.$\square$

We denote by $(G/H, \mathcal{O}_{G/H})$ the supermanifold defined by the holomorphic atlas constructed above. From the definition of transition functions between the charts $(gV, \mathcal{O}_{gV})$, we get a morphism $p : (G, \mathcal{O}_G) \to (G/H, \mathcal{O}_{G/H})$ with $p|_{gU} = p_{gU}$ for all $g \in G$. Now we can generalize Lemma 2.

ł *Let $W \subset G/H$ be an open set and $f \in \mathcal{O}_G(p_1^{-1}(W))$. Then $f \in \mathrm{Im}(p_2)$ if and only if f is $(H, \mathcal{O}_H)$-right invariant.* $\square$

We now define an action of the Lie supergroup $(G, \mathcal{O}_G)$ on the supermanifold $(G/H, \mathcal{O}_{G/H})$, which will be used in the proof of the main result of this section (Theorem 4). Denote by $i_g : (gV, \mathcal{O}_{G/H}) = (gS, \mathcal{O}_{gS}) \to (gU, \mathcal{O}_G)$ the natural embedding. Let α_{gV} be the following composition:

$$(G, \mathcal{O}_G) \times (gV, \mathcal{O}_{G/H}) \xrightarrow{\mathrm{id} \times i_g} (G, \mathcal{O}_G) \times (gU, \mathcal{O}_G) \xrightarrow{\nu} (G, \mathcal{O}_G) \xrightarrow{p} (G/H, \mathcal{O}_{G/H}). \quad (5)$$

Suppose $g_1V \cap g_2V \neq \emptyset$. We claim that $\alpha_{g_1V}|_{G\times g_1V\cap g_2V} = \alpha_{g_2V}|_{G\times g_1V\cap g_2V}$. Obviously, $(\alpha_{g_1V})_1|_{G\times(g_1V\cap g_2V)} = (\alpha_{g_2V})_1|_{G\times(g_1V\cap g_2V)}$, and so we only have to prove our equality for the second components of the morphisms α_{g_1V} and α_{g_2V}. Let $W \subset G/H$ be an open set and $f \in \mathcal{O}_{G/H}(W)$. It suffices to check that

$$(\mathrm{id}_2 \times (i_{g_1})_2)(\nu_2 \circ p_2(f)) = (\mathrm{id}_2 \times (i_{g_2})_2)(\nu_2 \circ p_2(f)). \quad (6)$$

From the associativity of the multiplication in $(G, \mathcal{O}_G)$ and from Lemma 3, it follows that

$$(\mathrm{id}_2 \times (\nu|_{G\times H})_2)(\nu_2 \circ p_2(f)) = (\mathrm{id}_2 \times (\mathrm{pr}_G^{G\times H})_2)(\nu_2 \circ p_2(f)).$$

Let $(O, \mathcal{O}_G)$ and $(K, \mathcal{O}_G)$ be such open sets in $(G, \mathcal{O}_G)$ that $O \times K \subset \nu_1^{-1}(p_1^{-1}(W))$. We assume that $(O, \mathcal{O}_G)$ is a coordinate neighborhood with even and odd coordinates (y_j) and $K = p_1^{-1}(N)$, where N is an open set in G/H. Denote by (t_i) a maximal independent system of monomials in y_j. By Lemma 2 we can write the function $\nu_2 \circ p_2(f)|_{O\times K}$ in the form $\sum t_i s_i$, where $s_i \in \mathcal{O}_G(K)$. We have in $O \times K \times H$

$$\sum t_i (\mathrm{pr}_G^{G\times H})_2(s_i) = (\mathrm{id}_2 \times (\mathrm{pr}_G^{G\times H})_2)(\sum t_i s_i) = (\mathrm{id}_2 \times (\mathrm{pr}_G^{G\times H})_2)(\nu_2 \circ p_2(f)) =$$

$$= (\mathrm{id}_2 \times (\nu|_{G\times H})_2)(\nu_2 \circ p_2(f)) = (\mathrm{id}_2 \times (\nu|_{G\times H})_2)(\sum t_i s_i) = \sum t_i (\nu|_{G\times H})_2(s_i).$$

Now the equality $(\mathrm{pr}_G^{G\times H})_2(s_i) = (\nu|_{G\times H})_2(s_i)$ follows from the independence of the system (t_i). In other words, we get that the functions s_i are $(H, \mathcal{O}_H)$-right invariant. By Lemma 3 there are functions $h_i \in \mathcal{O}_{G/H}(N)$, such that $p_2(h_i) = s_i$. Moreover,

$$(\mathrm{id}_2 \times (i_{g_1})_2)(\nu_2 \circ p_2(f)|_{O\times K}) = (\mathrm{id}_2 \times (i_{g_1})_2)(\sum t_i s_i) = \sum t_i (i_{g_1})_2(s_i) = \sum t_i (i_{g_1})_2(p_2(h_i)).$$

Now we use the trivial equality $p \circ i_g = \mathrm{id}$ for all $g \in G$.

$$\sum t_i (i_{g_1})_2(p_2(h_i)) = \sum t_i h_i.$$

Similarly, we obtain

$$(\mathrm{id}_2 \times (i_{g_2})_2)(\nu_2 \circ p_2(f)|_{O\times K}) = \sum t_i h_i.$$

So we have shown that

$$(\mathrm{id}_2 \times (i_{g_1})_2)(\nu_2 \circ p_2(f))|_{O\times N} = (\mathrm{id}_2 \times (i_{g_2})_2)(\nu_2 \circ p_2(f))|_{O\times N}.$$

This implies (6), and so we get a morphism α, such that $\alpha|_{G\times gV} = \alpha_{gV}$. Clearly, α is an action on the supermanifold G/H. We have proved the following theorem.

Theorem 4. *There exists a supermanifold $(G/H, \mathcal{O}_{G/H})$, such that the natural action of G on G/H induces a transitive action of $(G, \mathcal{O}_G)$ on $(G/H, \mathcal{O}_{G/H})$. The action of the Lie supergroup $(G, \mathcal{O}_G)$ on $(G/H, \mathcal{O}_{G/H})$ is given by (5).*$\square$

3. Stationary Lie subsupergroup

Let $\mu : (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$ be an action of a Lie supergroup $(G, \mathcal{O}_G)$ on a supermanifold $(M, \mathcal{O}_M)$ and let $\mathfrak{g}$ be the Lie superalgebra of the Lie supergroup $(G, \mathcal{O}_G)$. Denote by $\mu_x : (G, \mathcal{O}_G) \to (M, \mathcal{O}_M)$, $x \in M$, the composition of morphisms

$$(G, \mathcal{O}_G) \times (\mathrm{pt}, \mathbb{C}) \xrightarrow{id\times\widehat{x}} (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \xrightarrow{\mu} (M, \mathcal{O}_M),$$

where $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) : (\mathrm{pt}, \mathbb{C}) \to (M, \mathcal{O}_M)$, $\widehat{x}_1(\mathrm{pt}) = x$, $\widehat{x}_2(f) = f(x)$, $f \in \mathcal{O}_M$. Also, let $\overline{l}_g : (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$, $g \in G$, be the composition of morphisms

$$(M, \mathcal{O}_M) \simeq (\mathrm{pt}, \mathbb{C}) \times (M, \mathcal{O}_M) \xrightarrow{\widehat{g}\times id} (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \xrightarrow{\mu} (M, \mathcal{O}_M),$$

where $\widehat{g}$ was defined in Section 1.

ł *We have $\mathrm{ev}_x \circ \widetilde{\mu}(X) = (\mathrm{d}\mu_x)_e(\mathrm{ev}_e(X))$, $X \in \mathfrak{g}$. The action μ is transitive if and only if μ_x is a submersion at $e \in G$ for all $x \in M$.*

Proof. The second assertion follows from the first one. Let us prove the first assertion. By definition we get

$$\mathrm{ev}_x(\widetilde{\mu}(X))(f) = (\widetilde{\mu}(X)(f))(x), \quad (\mathrm{d}\mu_x)_e(\mathrm{ev}_e(X))(f) = (X((\mu_x)_2(f)))(e)$$

for all $f \in (\mathcal{O}_M)_x$. Therefore,

$$\begin{aligned}\mathrm{ev}_x(\widetilde{\mu}(X))(f) = \ & (\mathrm{id}_2 \times \widehat{x}_2)((\varepsilon_2 \times \mathrm{id}_2)((X \oplus 0) \circ \mu_2(f))) = \\ & (\varepsilon_2 \times \mathrm{id}_2)((\mathrm{id}_2 \times \widehat{x}_2)((X \oplus 0) \circ \mu_2(f))) = \\ & \varepsilon_2(X((\mathrm{id}_2 \times \widehat{x}_2) \circ \mu_2(f))) = \varepsilon_2(X((\mu_x)_2(f))) = \ (\mathrm{d}\mu_x)_e(X_e)(f).\square\end{aligned}$$

The following lemma is a consequence of the axioms of action.

ł *We have $\mu_x \circ r_g = \mu_{gx}$, $\mu_x \circ l_g = \overline{l}_g \circ \mu_x$.* $\square$

Consider a superdomain $(V, \mathcal{O}_M) \subset (M, \mathcal{O}_M)$, such that x is contained in V, with even coordinates (y_i) and odd coordinates (η_j). Suppose that x has the coordinates $y_i = 0$, $\eta_j = 0$.

Denote by $(U, \mathcal{O}_G)$ a superdomain in $((\mu_x)_1^{-1}(V), \mathcal{O}_G)$ with even coordinates (x_s) and odd coordinates (ξ_t). Let the morphism $\mu_x|(U, \mathcal{O}_G)$ be given by the equations:

$$(\mu_x)_2(y_i) = \phi^i_U(x_s, \xi_t), \quad (\mu_x)_2(\eta_j) = \psi^j_U(x_s, \xi_t).$$

Denote by $\mathcal{I}_U$ the sheaf of ideals in the structure sheaf of U defined as follows: if $U \cap (\mu_x)_1^{-1}(V) \neq \emptyset$ then $\mathcal{I}_U$ is generated by the functions $\phi^i_U(x_s, \xi_t)$, $\psi^j_U(x_s, \xi_t)$, otherwise $\mathcal{I}_U := \mathcal{O}_G|_U$. The sheafs $\mathcal{I}_{U_1}$ and $\mathcal{I}_{U_2}$ coincide on the intersection $U_1 \cap U_2$. Therefore there is a sheaf of ideals $\mathcal{I}$, such that $\mathcal{I}|_U = \mathcal{I}_U$. As usual, denote by G_x the stationary subgroup of the action μ_1 at x and consider the ringed space $(G_x, \mathcal{O}_{G_x})$, where $\mathcal{O}_{G_x} = (\mathcal{O}_G/\mathcal{I})|_{G_x}$.

If μ is transitive then $(G_x, \mathcal{O}_{G_x})$ is a subsupermanifold of $(G, \mathcal{O}_G)$. Indeed, by Lemma 4 the morphism μ_x is a submersion at e. From Lemma 4 it follows that μ_x is a submersion at every point $g \in G$. By the definition of a submersion, there is a coordinate neighborhood $(W, \mathcal{O}_G)$ of $g \in G_x$ with coordinates $(x_i; \xi_j)$, $i = 1, \dots, p$, $j = 1, \dots, q$ and a coordinate neighborhood of x with coordinates $(y_a; \eta_b)$, $a = 1, \dots, k$, $b = 1, \dots, l$, such that $\mu_x|(W, \mathcal{O}_G)$ is given by the following formulas:

$$(\mu_x)_2(y_a) = x_a, \;\; a = 1, \dots, k, \quad (\mu_x)_2(\eta_b) = \xi_b, \;\; b = 1, \dots, l. \tag{7}$$

Without loss of generality assume that the point x is given by the system of equations $y_a = 0$, $a = 1, \dots, k$, $\eta_b = 0$, $b = 1, \dots, l$. Then $(G_x \cap W, \mathcal{O}_{G_x})$ is isomorphic to a superdomain with coordinates x_i, $i = k+1, \dots, p$, ξ_j, $j = l+1, \dots, q$.

It is not hard to prove that $\iota^2 = \mathrm{id}$. Indeed,

$$\nu \circ (\mathrm{id} \times \nu) \circ (\iota^2, \iota, \mathrm{id}) = \nu \circ (\iota^2, \nu \circ (\iota, \mathrm{id})) = \nu \circ (\iota^2, \varepsilon) = \iota^2$$

and

$$\nu \circ (\mathrm{id} \times \nu) \circ (\iota^2, \iota, \mathrm{id}) = \nu \circ (\nu \times \mathrm{id}) \circ ((\iota, \mathrm{id}) \times \mathrm{id}) \circ (\iota, \mathrm{id}) = \nu \circ (\varepsilon \times \mathrm{id}) \circ (\iota, \mathrm{id}) = \mathrm{id}\,.$$

We will use this conclusion in the proof of the following theorem.

Theorem 5. *Suppose $(G_x, \mathcal{O}_{G_x}) \subset (G, \mathcal{O}_G)$ is a subsupermanifold. Then $(G_x, \mathcal{O}_{G_x})$ is a Lie subsupergroup of $(G, \mathcal{O}_G)$.*

Proof. We must show that $(G_x, \mathcal{O}_{G_x})$ is ν-invariant and ι-invariant. We check first that

$$(G_x, \mathcal{O}_{G_x}) \;\; \textit{is } \nu\textit{-invariant.} \tag{8}$$

Obviously, $\nu_1(G_x, G_x) = G_x$. Denote by $\mathcal{J}$ the sheaf of ideals corresponding to the subsupermanifold $(G_x, \mathcal{O}_{G_x}) \times (G_x, \mathcal{O}_{G_x})$ of the supermanifold $(G, \mathcal{O}_G) \times (G, \mathcal{O}_G)$. We have to prove that $\nu_2(\mathcal{I}) \subset \mathcal{J}$. The functions $\phi^i_U(x_s, \xi_t)$, $\psi^j_U(x_s, \xi_t)$ generate the ideal sheaf $\mathcal{I}|_U$ and $\mathcal{I}_y \neq (\mathcal{O}_G)_y$ only for $y \in G_x \subset W := (\mu_x)_1^{-1}(V)$. Therefore it is sufficient to prove that $\nu_2(\phi^i_U(x_s, \xi_t))|_{W \times W} \subset \mathcal{J}|_{W \times W}$ and $\nu_2(\psi^j_U(x_s, \xi_t))|_{W \times W} \subset \mathcal{J}|_{W \times W}$. By the definition of the functions $\phi^i_U(x_s, \xi_t)$ and $\psi^j_U(x_s, \xi_t)$, we get $\nu_2(\phi^i_U(x_s, \xi_t)) = \nu_2((\mu_x)_2(y_i))$, $\nu_2(\psi^j_U(x_s, \xi_t)) = \nu_2((\mu_x)_2(\eta_j))$. From the axioms of action it follows that $\mu_x \circ \nu = \mu \circ (\mathrm{id} \times \mu_x)$. Thus it suffices to prove that $(\mathrm{id}_2 \times (\mu_x)_2) \circ \mu_2(y_i)|_{W \times W} \subset \mathcal{J}|_{W \times W}$ and $(\mathrm{id}_2 \times (\mu_x)_2) \circ \mu_2(\eta_j)|_{W \times W} \subset \mathcal{J}|_{W \times W}$.

Denote by $\mathrm{pr}_i : (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ the projection on the i-th factor and by $\widetilde{(\mathrm{pr}_i)_2}(\mathcal{I})$ the sheaf of ideals generated by $(\mathrm{pr}_i)_2(\mathcal{I})$, $i = 1, 2$. We have $\mathcal{J} = \widetilde{(\mathrm{pr}_1)_2}(\mathcal{I}) +$

$\widetilde{(\mathrm{pr}_2)_2}(\mathcal{I})$. Further, $[(\mathrm{id}_2 \times (\mu_x)_2) \circ \mu_2(y_i)]|_{W\times W} = (\mathrm{id}_2 \times (\mu_x)_2)[\mu_2(y_i)|_{W\times V}]$. Using the definition of $\mathcal{I}$, we get:

$$(\mathrm{id}_2 \times (\mu_x)_2)[\mu_2(y_i)|_{W\times V}] + \widetilde{(\mathrm{pr}_2)_2}(\mathcal{I})|_{W\times W} = (\mathrm{pr}_1)_2((\mu_x)_2(y_i))|_{W\times W} + \widetilde{(\mathrm{pr}_2)_2}(\mathcal{I})|_{W\times W}.$$

Now, from $(\mathrm{pr}_1)_2((\mu_x)_2(y_i))|_{W\times W} \in \widetilde{(\mathrm{pr}_1)_2}(\mathcal{I})|_{W\times W}$ it follows that

$$(\mathrm{id}_2 \times (\mu_x)_2) \circ \mu_2(y_i)|_{W\times W} \in \widetilde{(\mathrm{pr}_1)_2}(\mathcal{I})|_{W\times W} + \widetilde{(\mathrm{pr}_2)_2}(\mathcal{I})|_{W\times W} = \mathcal{J}|_{W\times W}$$

and, by the same argument, $(\mathrm{id}_2 \times (\mu_x)_2) \circ \mu_2(\eta_j)|_{W\times W} \subset \mathcal{J}|_{W\times W}$. This completes the proof of (8).

It remains to check that

$$(G_x, \mathcal{O}_{G_x}) \ \ \textit{is } \iota\textit{-invariant.}$$

Since the inclusion $\iota_1(G_x) \subset G_x$ is obvious, we must prove that $\iota_2(\mathcal{I}) \subset \mathcal{I}$ or, in terms of generators, $\iota_2(\phi^i_U(x_s, \xi_t)) \in \mathcal{I}$ and $\iota_2(\psi^i_U(x_s, \xi_t)) \in \mathcal{I}$.

By definition of the supermanifold $(G_x, \mathcal{O}_{G_x})$, the following diagram is commutative:

$$\begin{array}{ccc}
(G_x, \mathcal{O}_{G_x}) & \xrightarrow{\mathrm{pr}_x} & (x, \mathbb{C}) \\
\Big\| & & \Big\downarrow \widehat{x} \\
(G_x, \mathcal{O}_{G_x}) & \xrightarrow{\mu_x} & (M, \mathcal{O}_M)
\end{array} .$$

We will rather use the commutativity of the next diagram:

$$\begin{array}{ccc}
(\iota_1(G_x), \mathcal{O}_{\iota_1(G_x)}) & \xrightarrow{\mathrm{pr}_x} & (x, \mathbb{C}) \\
\Big\| & & \Big\downarrow \widehat{x} \\
(\iota_1(G_x), \mathcal{O}_{\iota_1(G_x)}) & \xrightarrow{\mu_x} & (M, \mathcal{O}_M)
\end{array} . \tag{9}$$

To show that (9) is commutative, note that, by the inverse element axiom of a Lie supergroup and by the axioms of an action, $\mu \circ (\iota, \mu_x) = \widehat{x} \circ \mathrm{pr}_x$ and, in particular, $\mu \circ (\iota, \mu_x)|_{G_x} = (\widehat{x} \circ \mathrm{pr}_x)|_{G_x}$. Using the equality $\mu_x|_{G_x} = \widehat{x}$ and the definition of the morphism μ_x, we obtain the commutative diagram:

$$\begin{array}{ccc}
(G_x, \mathcal{O}_{G_x}) & \xrightarrow{\mathrm{pr}_x} & (x, \mathbb{C}) \\
{\scriptstyle (\iota, \widehat{x})}\Big\downarrow & & \Big\downarrow \widehat{x} \\
(\iota_1(G_x), \mathcal{O}_{\iota_1(G_x)}) \times (x, \mathbb{C}) & \xrightarrow{\mu_x} & (M, \mathcal{O}_M)
\end{array} .$$

Since $(\iota, \widehat{x})$ is an isomorphism, we get the commutativity of (9).

Denote by $\mathcal{I}_{\iota(G_x)}$ the sheaf of ideals in $\mathcal{O}_G$ corresponding to the subsupermanifold $(\iota_1(G_x), \mathcal{O}_{\iota_1(G_x)}) \subset (G, \mathcal{O}_G)$. Using (9), we have:

$$\phi^i_U(x_s, \xi_t) + \mathcal{I}_{\iota(G_x)} = \mu_x(y_i) + \mathcal{I}_{\iota(G_x)} = y_i(x) + \mathcal{I}_{\iota(G_x)} = 0 + \mathcal{I}_{\iota(G_x)} = \mathcal{I}_{\iota(G_x)}.$$

Therefore $\phi^i_U(x_s, \xi_t) \in \mathcal{I}_{\iota(G_x)}$. Similarly, $\psi^i_U(x_s, \xi_t) \in \mathcal{I}_{\iota(G_x)}$. It follows that $\mathcal{I} \subset \mathcal{I}_{\iota(G_x)} = \iota_2(\mathcal{I})$, and the equality $\iota^2 = \mathrm{id}$ implies that $\mathcal{I} = \mathcal{I}_{\iota(G_x)}$.$\square$

We have seen that for a transitive action of $(G, \mathcal{O}_G)$ on $(M, \mathcal{O}_M)$ the ringed space $(G_x, \mathcal{O}_{G_x})$ is a supermanifold, which is in fact a Lie subsupergroup of $(G, \mathcal{O}_G)$. This Lie subsupergroup will be called *the stationary Lie subsupergroup of* x. In the last section we will prove that any homogeneous supermanifold is isomorphic to a coset space of a Lie supergroup with the structure of a supermanifold introduced above.

4. The structure of a homogeneous space

Our goal here is the following theorem.

Theorem 6. *Let $(M, \mathcal{O}_M)$ be a $(G, \mathcal{O}_G)$-homogeneous supermanifold and let $(H, \mathcal{O}_H)$ be a stationary Lie subsupergroup of a point $x \in M$. Then there is an isomorphism*

$$\beta : (G/H, \mathcal{O}_{G/H}) \to (M, \mathcal{O}_M),$$

which is $(G, \mathcal{O}_G)$-equivariant in the sense that the following diagram is commutative:

$$\begin{array}{ccc} (G, \mathcal{O}_G) \times (G/H, \mathcal{O}_{G/H}) & \xrightarrow{\mathrm{id} \times \beta} & (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \\ {\scriptstyle \alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu} \\ (G/H, \mathcal{O}_{G/H}) & \xrightarrow{\beta} & (M, \mathcal{O}_M) \end{array} . \tag{10}$$

Proof. Let $\beta_1 : G/H \to M$, $gH \mapsto gx$, be the natural homeomorphism. Then the diagram

$$\begin{array}{ccc} (G, \mathcal{O}_G) & = & (G, \mathcal{O}_G) \\ {\scriptstyle p=(p_1,p_2)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu_x=((\mu_x)_1,(\mu_x)_2),} \\ (G/H, \mathcal{O}_{G/H}) & \xrightarrow{\beta_1} & (M, \mathcal{O}_M) \end{array} \tag{11}$$

is commutative. We will now construct an isomorphism of sheaves $\beta_2 : \mathcal{O}_M \to (\beta_1)_* \mathcal{O}_{G/H}$, such that $\beta = (\beta_1, \beta_2)$ is the required isomorphism.

As in Theorem 2, consider the product $(S, \mathcal{O}_S) \times (H, \mathcal{O}_H)$. By the axioms of action, the composition of morphisms

$$(S, \mathcal{O}_S) \times (H, \mathcal{O}_H) \xrightarrow{\nu|_{S \times H}} (U, \mathcal{O}_G) \xrightarrow{\mu_x} (M, \mathcal{O}_M).$$

can be written as $\mu_x \circ \nu|_{S \times H} = \mu \circ (\mathrm{id} \times \mu_x|_H) = \mu \circ (\mathrm{id} \times \widehat{x})$. The last equality follows from the fact that $\mu_x|_H = \widehat{x}$. As a result, we get

$$\mu_x \circ \nu|_{S \times H} = \mu_x \circ \mathrm{pr}_S^{S \times H} . \tag{12}$$

The differential $(\mathrm{d}\mu_x)_e$ is surjective, because the action μ is transitive. The differential $(\mathrm{d}\nu|_{S \times H})_{(e,e)}$ is nondegenerate, see Theorem 2. It is easy to see that $\dim(S, \mathcal{O}_S) = \dim(M, \mathcal{O}_M)$, and so we get that $(\mathrm{d}(\mu_x)|_S)_{(e)}$ is nondegenerate. By Theorem 1, we can assume that $\mu|_S$ is an isomorphism of $(S, \mathcal{O}_S)$ onto some superdomain $(V, \mathcal{O}_M) \subset (M, \mathcal{O}_M)$. By (12) we

get $(\mu_x)_2(\mathcal{O}_M|_V) = ((\nu|_{S\times H})^{-1})_2(\mathcal{O}_S)$. On the other hand, by the definition of p we have $p_2(\mathcal{O}_{G/H}|_{p_1(S\times H)}) = ((\nu|_{S\times H})^{-1})_2(\mathcal{O}_S)$ and therefore $p_2(\mathcal{O}_{G/H}|_{p_1(S\times H)}) = (\mu_x)_2(\mathcal{O}_M|_V)$.

Let $\widetilde{l}_g = \alpha \circ (\widehat{g} \times \mathrm{id})$. The following diagram is commutative by Lemma 4:

$$\begin{array}{ccc}
(p_1(S\times H), \mathcal{O}_{G/H}) & \xrightarrow{\widetilde{l}_g} & (p_1(gS\times H), \mathcal{O}_{G/H}) \\
{\scriptstyle p}\uparrow & & {\scriptstyle p}\uparrow \\
(U, \mathcal{O}_G) & \xrightarrow{l_g} & (gU, \mathcal{O}_G) \\
{\scriptstyle \mu_x}\downarrow & & {\scriptstyle \mu_x}\downarrow \\
(V, \mathcal{O}_M) & \xrightarrow{\bar{l}_g} & (gV, \mathcal{O}_M)
\end{array} \qquad (13)$$

By the commutativity of (13), we get

$$(\mu_x)_2(\mathcal{O}_M|_{gV}) = p_2(\mathcal{O}_{G/H}|_{p_1(gU)}), \quad \text{for all } g \in G.$$

Therefore, the sheaves $(\mu_x)_2(\mathcal{O}_M)$ and $p_2(\mathcal{O}_{G/H})$ coincide locally, and it follows that $p_2(\mathcal{O}_{G/H}) = (\mu_x)_2(\mathcal{O}_M)$. Define a morphism $\beta_2 : \mathcal{O}_M \to (\beta_1)_*(\mathcal{O}_{G/H})$ by

$$\beta_2(f) = p_2^{-1} \circ (\mu_x)_2(f).$$

Obviously, the morphism $\beta = (\beta_1, \beta_2)$ is an isomorphism and the diagram (11) is commutative.

It remains to prove that diagram (10) is also commutative. By the definition of β, we get locally $\beta = (\beta_1, \beta_2) = \mu_x \circ i_g$, where i_g is defined after Lemma 3. Hence $\mu_x \circ i_g \circ p = \mu_x$ and the axioms of action yield $\mu \circ (\mathrm{id} \times \mu_x) = \mu_x \circ \nu$. Thus we get locally:

$$\mu \circ (\mathrm{id} \times \beta) = \mu \circ (\mathrm{id} \times \mu_x \circ i_g) = \mu \circ (\mathrm{id} \times \mu_x) \circ (\mathrm{id} \times i_g) = \mu_x \circ \nu \circ (\mathrm{id} \times i_g).$$

$$\beta \circ \alpha = \mu_x \circ i_g \circ p \circ \nu \circ (\mathrm{id} \times i_g) = \mu_x \circ \nu \circ (\mathrm{id} \times i_g).$$

This implies the commutativity of the diagram (10).□

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Tver State University, Zhelyabova 33, 170 000 Tver, Russia
E-mail address: `VishnyakovaE@googlemail.com`